Published by Institute of Physics Publishing for SISSA
Received: *April 18, 2005*
Revised: *July 25, 2005*
Accepted: *September 14, 2005*
Published: *September 27, 2005*

U(1) gauge field theory on κ-Minkowski space

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Abstract: This study of U(1) gauge field theory on the kappa-deformed Minkowski spacetime extends previous work on gauge field theories on this type of noncommutative spacetime. We discuss in detail the properties of the Seiberg-Witten map and the resulting effective action for U(1) gauge theory with fermionic matter expanded in ordinary fields. We construct the conserved gauge current, fix part of the ambiguities in the Seiberg-Witten map and obtain an effective U(1) action invariant under the action of the undeformed Poincaré group.

Keywords: Gauge Symmetry, Non-Commutative Geometry.

JHEP09(2005)068

Contents

1. Introduction

In previous papers [1, 2] (gauge) field theories on the κ-Minkowski spacetime were constructed. All techniques necessary for such a construction were thoroughly discussed there. In this paper we concentrate on the U(1) gauge field theory on the κ-Minkowski spacetime. In the first order of expansion in a deformation parameter, we construct an effective U(1) gauge field theory, represented on commutative spacetime, using the $\star$-product formulation and the Seiberg-Witten map. We also construct a conserved gauge current. The appearance of the additional conserved current forces us to analyse ambiguities in the Seiberg-Witten map. A specific choice of the Seiberg-Witten map provides an effective U(1) action invariant under the action of the undeformed Poincaré group.

The paper is organized as follows: in the section 2 we briefly review some of the properties of the κ-Minkowski spacetime. In the section 3 the Seiberg-Witten map for gauge fields is constructed and using this result the field strength tensor is calculated. In the section 4 we construct the action for U(1) gauge theory with fermionic matter and analyse the effective action obtained by expansion in the deformation parameter. Finally, in the section 5 we construct the conserved gauge current, discuss ambiguities of the Seiberg-Witten map and construct an action invariant under the action of the undeformed Poincaré group.

2. κ-Minkowski spacetime

Algebraically, the $n+1$-dimensional κ-Minkowski spacetime can be introduced [1, 2] as a factor space of the algebra freely generated by coordinates $\hat{x}^\mu$ divided by the ideal generated by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho \,, \tag{2.1}$$

where

$$C_\rho^{\mu\nu} = a(\delta_n^\mu \delta_\rho^\nu - \delta_n^\nu \delta_\rho^\mu), \qquad \mu = 0, \ldots, n\,, \tag{2.2}$$

and the (formal) metric of the κ-Minkowski spacetime is $\eta^{\mu\nu} = diag(1, -1, \ldots, -1)$. A constant deformation vector a^μ of length a points to the n-th spacelike direction, $a^n = a$, and is related to the frequently used parameter κ as $a = 1/\kappa$. Latin indices denote undeformed dimensions, n is the deformed dimension and the Greek indices refer to all $n+1$ dimensions.

There exists an isomorphism between this abstract algebra and the algebra of functions of commuting variables equipped with a $\star$-product [3]. Our goal is to construct an effective field theory on the ordinary spacetime. Therefore we work in the $\star$-product formalism. The symmetric $\star$-product for the κ-Minkowski spacetime,[1] up to first order in the deformation parameter, is given by

$$\begin{aligned} f(x) \star g(x) &= f(x)g(x) + \frac{i}{2} C_\lambda^{\mu\nu} x^\lambda \partial_\mu f(x) \partial_\nu g(x) \\ &= f(x)g(x) + \frac{ia}{2} x^j \left(\partial_n f(x) \partial_j g(x) - \partial_j f(x) \partial_n g(x)\right). \end{aligned} \tag{2.3}$$

From (2.3) we have

$$[x^n \stackrel{\star}{,} x^j] = x^n \star x^j - x^j \star x^n = iax^j\,, \qquad [x^i \stackrel{\star}{,} x^j] = 0\,; \qquad i, j = 0, 1, \ldots, n-1. \tag{2.4}$$

In order to construct derivatives on this space, we impose several formal conditions that these derivatives should fulfil. The two most critical requirements are first, that the derivatives should be consistent with the commutation relations (2.4) [5], and second, that they should be antihermitean under the integral which we later introduce to define the effective action. The first requirement, together with the demand that derivatives should commute among themselves, does not fix the derivatives sufficiently [6]. If we add an additional requirement that derivatives should have a vector-like transformation law under κ-deformed Lorentz transformations,[2] we obtain an almost [6] unique solution which we call Dirac derivative

$$\begin{aligned} [D_n^* \stackrel{\star}{,} x^j] &= -iaD^{j*}\,, \\ [D_n^* \stackrel{\star}{,} x^n] &= \sqrt{1 + a^2 D^{\mu*} D_\mu^*}\,, \\ [D_i^* \stackrel{\star}{,} x^j] &= \eta_i{}^j \left(-iaD_n^* + \sqrt{1 + a^2 D^{\mu*} D_\mu^*}\right), \\ [D_i^* \stackrel{\star}{,} x^n] &= 0\,. \end{aligned} \tag{2.5}$$

The particular choice of derivative we use [1, 6] is convenient since in this basis the κ-Poincare algebra remains undeformed and the deformation is in the coalgebra sector. This

[1] There is a standard $\star$-product for Lie algebras [4].

[2] In this paper we omit the representation of the generators of κ-Lorentz transformation because they are not essential for the problem at hand. Note that the κ-deformed Poincaré algebra was first introduced in ref. [7].

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basis was also used in [8], where it was called "classical" basis. Using relations (2.5) we calculate the representation of the derivatives D^*_μ on ordinary functions,

$$\begin{aligned} D^*_n f(x) &= \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\partial_j\partial^j\right) f(x)\,, \\ D^*_i f(x) &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\partial_i f(x)\,. \end{aligned} \tag{2.6}$$

Formulae (2.6) deliver the representation of D^*_μ derivatives in terms of the usual partial derivatives ∂_μ.

With this choice of derivatives the first condition is fulfilled. In order to fulfil the second one as well, we need $\tilde{D}^*_\mu$ derivatives that are antihermitean under the integral with a measure μ

$$\int d^{n+1}x\mu(x)\;\bar{f}\star\tilde{D}^*_\alpha g = -\int d^{n+1}x\mu(x)\overline{\tilde{D}^*_\alpha f}\star g\,. \tag{2.7}$$

This demand will become clearer in section 4. The derivatives $\tilde{D}^*_\alpha$ are obtained by substituting

$$\partial_i \longrightarrow \tilde{\partial}_i = \partial_i + \frac{\partial_i\mu}{2\mu}\,, \qquad \partial_n \longrightarrow \tilde{\partial}_n = \partial_n$$

in (2.6). The representation of these improved derivatives on functions is

$$\begin{aligned} \tilde{D}^*_n f(x) &= \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{ia\partial_n^2}\tilde{\partial}_j\tilde{\partial}^j\right) f(x)\,, \\ \tilde{D}^*_i f(x) &= \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\tilde{\partial}_i f(x)\,. \end{aligned} \tag{2.8}$$

The Leibniz rules for $\tilde{D}^*_\mu$ derivatives (as well as for the D^*_μ derivatives) are non-trivial as a consequence of (2.5)

$$\begin{aligned} \tilde{D}^*_n\left(f(x)\star g(x)\right) &= (D^*_n f(x))\star(e^{-ia\partial_n}g(x)) + (e^{ia\partial_n}f(x))\star(\tilde{D}^*_n g(x)) - \\ &\quad - ia\left(D^*_j e^{ia\partial_n}f(x)\right)\star(\tilde{D}^{j*}g(x))\,, \\ \tilde{D}^*_i\left(f(x)\star g(x)\right) &= (D^*_i f(x))\star(e^{-ia\partial^*_n}g(x)) + f(x)\star(\tilde{D}^*_i g(x))\,. \end{aligned} \tag{2.9}$$

In ref. [6] it has been shown that the fundamental object to construct geometrical theories on κ-deformed space-times is the exterior differential d, which is nilpotent and has an undeformed Leibniz rule. Furthermore, it has been shown that d can be decomposed as d= $\xi^{*\mu}D^*_\mu$ using a set of n+1 one-forms $\xi^{*\mu}$ dual to the Dirac derivative. These one-forms (and all forms in the deRham complex) are derivative valued, i.e. they have non-trivial commutation relations with coordinates, as a consequence of demanding that there are only $n+1$ one-forms in $n+1$-dimensional κ-deformed space-time. In the present paper, we go beyond the framework of ref. [6] in requiring from the start the anti-hermiticity of Dirac derivatives under an integral. The shift from ∂_j to $\tilde{\partial}_j$ does not affect the conclusions of ref. [6], there exists a set of one-forms $\tilde{\xi}^{*\mu}$ dual to the shifted Dirac derivatives $\tilde{D}^*_\mu$ such that the exterior differential d= $\tilde{\xi}^{*\mu}\tilde{D}^*_\mu$ is nilpotent with an undeformed Leibniz rule.

3. Gauge theories and the Seiberg-Witten map

Gauge theories on the κ-Minkowski spacetime have two new properties, as a consequence of noncommutativity, see [2] for details. A gauge field is both enveloping algebra-valued and derivative-valued. Having an enveloping algebra-valued gauge field leads to a theory with infinitely many degrees of freedom. The way out of this unphysical situation is provided in terms of the Seiberg-Witten map [9]. Using this map one can express noncommutative variables (gauge parameter, fields) in terms of commutative ones and in this way retain the same number of degrees of freedom as in the commutative case (where the degrees of freedom are Lie algebra-valued).

Explicit solutions of the SW map for the gauge parameter and the matter field are constructed[3] from the assumption that

$$\delta_\alpha \psi = i\Lambda_\alpha(x) \star \psi \tag{3.1}$$

and that this is a gauge transformation

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{\alpha\times\beta}\psi\,. \tag{3.2}$$

Up to first order in a, the solutions[4] are [2]

$$\Lambda_\alpha = \alpha - \frac{1}{4}C_\lambda^{\rho\sigma}x^\lambda\{A_\rho^0, \partial_\sigma\alpha\}\,, \tag{3.3}$$

$$\psi = \psi^0 - \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda A_\rho^0\partial_\sigma\psi^0 + \frac{i}{8}C_\lambda^{\rho\sigma}x^\lambda[A_\rho^0, A_\sigma^0]\psi^0\,. \tag{3.4}$$

In the following we concentrate on the derivative-valued gauge fields. Since we are using the modified derivatives $\tilde{D}_\mu^*$, we have to modify the solutions of the Seiberg-Witten map for the gauge field V_μ given in [2] as well. The covariant derivative $\tilde{\mathcal{D}}_\mu = \tilde{D}_\mu^* - i\tilde{V}_\mu$ is defined by its transformation law

$$\delta_\alpha\left(\tilde{\mathcal{D}}_\mu\psi(x)\right) = i\Lambda_\alpha(x)\star\tilde{\mathcal{D}}_\mu\psi(x)\,. \tag{3.5}$$

From (3.5), it follows that the trasformation law for the gauge field $\tilde{V}_\mu$ is

$$\delta_\alpha\tilde{V}_\mu\star\psi = \tilde{D}_\mu^*\left(\Lambda_\alpha\star\psi\right) - \Lambda_\alpha\star\left(\tilde{D}_\mu^*\psi\right) + i\Lambda_\alpha\star\tilde{V}_\mu\star\psi - i\tilde{V}_\mu\star\left(\Lambda_\alpha\star\psi\right). \tag{3.6}$$

Since $\tilde{D}_\mu^*$ derivatives have non-trivial Leibniz rules (2.9), we have to treat the first term on the right-hand side with much care. It is convenient to separate the n-th and the i-th components of (3.6).

First, we look at the i-th component. Using the Leibniz rule for $\tilde{D}_i^*$ (2.9), we obtain

$$\delta_\alpha\tilde{V}_i\star\psi = (D_i^*\Lambda_\alpha)\star\left(e^{-ia\partial_n}\psi\right) + i\Lambda_\alpha\star\tilde{V}_i\star\psi - i\tilde{V}_i\star\left(\Lambda_\alpha\star\psi\right). \tag{3.7}$$

[3] In this section we construct solutions for the non-abelian gauge theory, although starting form the next section we use solutions for U(1) gauge theory only.

[4] The Lie algebra-valued field A_μ^0 has the usual transformation property $\delta_\alpha A_\mu^0 = \partial_\mu\alpha - i[A_\mu^0, \alpha]$, and the matter field ψ^0 transforms as $\delta_\alpha\psi^0 = i\alpha\psi^0$.

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In order to solve this equation, we allow derivative-valued gauge field components $\tilde{V}_i$ as we did in [2]. Inserting the ansatz

$$\tilde{V}_i = V_i \star e^{-ia\partial_n} \tag{3.8}$$

in (3.7) leads (after using $e^{-ia\partial_n}(f \star g) = (e^{-ia\partial_n} f) \star e^{-ia\partial_n} g$ and omitting $e^{-ia\partial_n}\psi$ on the right-hand side) to

$$\delta_\alpha V_i = (D_i^* \Lambda_\alpha) + i\Lambda_\alpha \star V_i - iV_i \star \left(e^{-ia\partial_n}\Lambda_\alpha\right). \tag{3.9}$$

We solve this equation perturbatively, i.e., we expand the V_i field in the deformation parameter a:

$$V_i = V_i^0 + V_i^1 + \cdots$$

and use the solution for the gauge parameter Λ_α (3.3). Up to first order the solution for $\tilde{V}_i$ is[5]

$$\tilde{V}_i = A_i^0 - iaA_i^0\partial_n - \frac{ia}{2}\partial_n A_i^0 - \frac{a}{4}\{A_n^0, A_i^0\} + \frac{1}{4}C^{\rho\sigma}_{\lambda} x^\lambda \left(\{F^0_{\rho i}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A_i^0\}\right). \tag{3.10}$$

One notices that this solution is the same as in [2]. This is to be expected, because the $\tilde{V}_i$ field is only ∂_n derivative-valued and ∂_n is not modified.

Next, we look at the n-th component of equation (3.6). Using the Leibniz rule for $\tilde{D}_n^*$ (2.9) leads to

$$\begin{aligned} \delta_\alpha \tilde{V}_n \star \psi \;=\; & (D_n^*\Lambda_\alpha) \star e^{-ia\partial_n}\psi + \left((e^{ia\partial_n} - 1)\Lambda_\alpha\right) \star \tilde{D}_n^*\psi - \\ & - ia(D_j^* e^{ia\partial_n}\Lambda_\alpha) \star \tilde{D}^{j*}\psi + i\Lambda_\alpha \star \tilde{V}_n \star \psi - i\tilde{V}_n \star (\Lambda_\alpha \star \psi). \end{aligned} \tag{3.11}$$

We make the following ansatz:

$$\tilde{V}_n = V_{n1} \star e^{-ia\partial_n} + V_2 \star \tilde{D}_n^* + V^l_{n3} \star \tilde{D}_l^* \tag{3.12}$$

and insert it in equation (3.11). Collecting terms proportional to $\star e^{-ia\partial_n}\psi$, $\star\tilde{D}_n^*\psi$ and $\star\tilde{D}_l^*\psi$, we obtain transformation laws for the field components V_{n1}, V_2 and V^l_{n3}, respectively:

$$\begin{aligned} \delta_\alpha V_{n1} \;=\; & (D_n^*\Lambda_\alpha) + i\Lambda_\alpha \star V_{n1} - iV_{n1} \star (e^{-ia\partial_n}\Lambda_\alpha) - \\ & - iV_2 \star (D_n^*\Lambda_\alpha) - iV^l_{n3} \star (D_l^*\Lambda_\alpha), \end{aligned} \tag{3.13}$$

$$\delta_\alpha V_2 = ((e^{ia\partial_n} - 1)\Lambda_\alpha) + i\Lambda_\alpha \star V_2 - iV_2 \star (e^{ia\partial_n}\Lambda_\alpha), \tag{3.14}$$

$$\delta_\alpha V^l_{n3} = -ia(\partial^{l*}\Lambda_\alpha) + i\Lambda_\alpha \star V^l_{n3} - iV^l_{n3} \star \Lambda_\alpha - aV_2 \star (\partial^{l*}\Lambda_\alpha). \tag{3.15}$$

Up to first order in a the solutions for these equations are

$$\begin{aligned} V_{n1} &= A_n^0 - \frac{a}{2}\left(i\partial_j A^{0j} + A_j^0 A^{0j}\right) + \frac{1}{4}C^{\rho\sigma}_{\lambda} x^\lambda \left(\{F^0_{\rho n}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_n\}\right), \\ V_2 &= iaA_n^0\,, \\ V^j_{n3} &= -iaA^{0j}\,, \end{aligned}$$

[5]The field strength $F^0_{\mu\nu}$ is the usual field strength of the undeformed theory; $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu]$.

and

$$\tilde{V}_n = A^0_n - iaA^{0j}\tilde{\partial}_j - \frac{ia}{2}\partial_j A^{0j} - \frac{a}{2}A^0_j A^{0j} + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(\{F^0_{\rho n}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma A^0_n\}\Big). \qquad (3.16)$$

Comparing this solution with the solution for V_n in [2], we see that the only difference is the term $-iaA^{0j}\tilde{\partial}_j$ as a consequence of modifying the derivatives.

As the next step we construct the field-strength tensor. It is defined as

$$\mathcal{F}_{\mu\nu} = i[\tilde{\mathcal{D}}_\mu \overset{\star}{,} \tilde{\mathcal{D}}_\nu]\,. \qquad (3.17)$$

Applying this to the field ψ gives

$$\mathcal{F}_{ij} = \Big((D^*_i V_j) - (D^*_j V_i) - iV_i \star (e^{-ia\partial_n} V_j) + iV_j \star (e^{-ia\partial_n} V_i)\Big) \star e^{-2ia\partial_n}, \qquad (3.18)$$

$$\mathcal{F}_{nj} = F_{nj1} \star e^{-2ia\partial_n} + F_{nj2} \star e^{-ia\partial_n}\tilde{D}^*_n + F^l_{nj3} \star e^{-ia\partial_n}\tilde{D}^*_l, \qquad (3.19)$$

where

$$\begin{aligned}
F_{nj1} &= (D^*_n V_j) - (D^*_j V_{n1}) - iV_{n1} \star (e^{-ia\partial_n} V_j) + iV_j \star (e^{-ia\partial_n} V_{n1}) \\
&\quad - iV^l_{n3} \star (D^*_l V_j) - iV_{n2} \star (D^*_n V_j)\,, \\
F_{nj2} &= ((e^{ia\partial_n} - 1))V_j - (D^*_j V_{n2}) - iV_{n2} \star (e^{ia\partial_n} V_j) + iV_j \star (e^{-ia\partial_n} V_{n2})\,, \\
F^l_{nj3} &= -ia(\partial^{l*} V_j) - (D^*_j V^l_{n3}) - aV_{n2} \star (\partial^{l*} V_j) - iV^l_{n3} \star V_j + V_j \star (e^{-ia\partial_n} V^l_{n3})\,.
\end{aligned}$$

It is obvious that the $\mathcal{F}_{\mu\nu}$ tensor is derivative valued. Therefore, we use the same procedure as in [2], namely, we split $\mathcal{F}_{\mu\nu}$ into the curvature-like terms and torsion-like terms[6]

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu}\tilde{\mathcal{D}}_\rho + \cdots + T^{\rho_1 \ldots \rho_l}_{\mu\nu} : \tilde{\mathcal{D}}_{\rho_1} \ldots \tilde{\mathcal{D}}_{\rho_l} : + \cdots\,. \qquad (3.20)$$

Expanding (3.18) and (3.19) up to first order in a and rewriting them in this form gives

$$\begin{aligned}
F_{ij} &= F^0_{ij} - ia\mathcal{D}^0_n F^0_{ij} + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(2\{F^0_{\rho i}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{ij}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{ij}\}\Big)\,, \\
T^\mu_{ij} &= -2ia\delta^\mu_n F^0_{ij}\,, \\
F_{nj} &= F^0_{nj} - \frac{ia}{2}\mathcal{D}^{\mu 0} F^0_{\mu j} + \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\Big(2\{F^0_{\rho n}, F^0_{\sigma j}\} + \{\mathcal{D}^0_\rho F^0_{nj}, A^0_\sigma\} - \{A^0_\rho, \partial_\sigma F^0_{nj}\}\Big)\,, \\
T^\mu_{nj} &= -ia\eta^{\mu l} F^0_{lj} - ia\delta^\mu_n F^0_{nj}\,. \qquad (3.21)
\end{aligned}$$

These results are the same as in ref. [2]. Actually, we have checked this up to second order in a, and because of the structure of equations (3.18), (3.19) and (2.9), we expect that this result holds to all orders in a. In the action for the gauge field, only curvature-like terms are used and from equations (3.21) we see that the modification of the derivatives does not affect the action.

In this paper we are interested in a *constructive* approach to formulating gauge theories. Due to this constructive approach we cannot provide a decisive answer whether the gauge theory presented here can also be formulated geometrically, with the gauge fields as components of a one-form or connection. However, we strongly assume that such geometrical formulation is not only possible but is the "proper" formulation of such a noncommutative gauge theory.

[6] Note that torsion-like terms are defined as the coefficients of modified covariant derivatives.

4. Action

In order to construct the action, we need an integral with the trace property. This is essential both for the formulation of the variational principle and for the gauge invariance of the action. We defined such an integral in the $\star$-product formalism, see ref. [1]. There we used the usual definition of an integral of functions of commuting variables and introduced a measure function to implement the trace property:

$$\int d^{n+1}x \; \mu(x)(f \star g) = \int d^{n+1}x \; \mu(x)(g \star f)\,. \tag{4.1}$$

Note that $\mu(x)$ is not $\star$-multiplied with the other functions, it is a part of the volume element. From (4.1), it follows that

$$\partial_n \mu(x) = 0\,, \qquad x^j \partial_j \mu(x) = -n\mu(x)\,. \tag{4.2}$$

The measure function is x^n independent and does not depend on the deformation parameter a either. In addition, the measure function is singular at zero and is not unique [10]. However, after defining the lagrangian density in such a way that it vanishes at zero, we can choose a positive-definite measure function. Note also that the explicit form of $\mu(x)$ is not required in any of the subsequent calculations. We only use relations (4.2), and therefore non-uniqueness of the solution for $\mu(x)$ does not affect our results.[7]

With this integral we define the action as follows:

$$S = \int d^{n+1}x \; \mu(x)\mathcal{L}\,,$$

where $\mathcal{L}$ is the lagrangian density. Since we saw that the measure function $\mu(x)$ does not vanish in the limit $a \to 0$, we have to define the lagrangian density such that

$$\lim_{a\to 0} \mu(x)\mathcal{L} = \mathcal{L}^0\,.$$

Here $\mathcal{L}$ is the effective lagrangian density expanded in powers of the deformation parameter a, and $\mathcal{L}^0$ is the lagrangian density of the corresponding undeformed field theory. Although this construction may appear rather arbitrary, we find that imposing such a "good" limit $a \to 0$ is an important physical requirement.

To this end, we now concentrate on U(1) gauge theory coupled to fermions. First, we analyse the action for matter fields. It should be the gauge covariant version of the action for the fermion matter fields defined in ref. [1], so the first guess would be

$$S_m = \int d^{n+1}x \; \mu(x) \left(\bar{\tilde{\psi}} \star (i\gamma^\mu \tilde{D}^*_\mu + \gamma^\mu \tilde{V}_\mu \star - m)\tilde{\psi} \right). \tag{4.3}$$

We have chosen the symbol $\tilde{\psi}$ instead of ψ for later convenience. Using the variational principle

$$\frac{\delta}{\delta g(x)} \int d^{n+1}x \; \mu(x) \; f \star g \star h = \frac{\delta}{\delta g(x)} \int d^{n+1}x \; \mu(x) \; g(h \star f) = \mu(x) \; (h \star f) \tag{4.4}$$

[7] An alternative way of constructing the measure function is using a map which connects the κ-Minkowski spacetime coordinates and the spacetime coordinates of the canonical noncommutative spacetime [11].

from the action (4.3), we obtain the equation of motion for the matter field $\tilde{\psi}$:

$$\mu(x)(i\gamma^\mu\tilde{\mathcal{D}}^*_\mu - m)\tilde{\psi} = 0\,. \tag{4.5}$$

It is obvious that this equation does not have the proper classical limit. In ref. [1] this problem was solved by rescaling the field $\tilde{\psi}$

$$\tilde{\psi} \to \mu^{-1/2}\psi\,. \tag{4.6}$$

Unfortunately, this rescaling is not fully compatible with the Seiberg-Witten map. Namely, if $\delta_\alpha\psi = i\Lambda_\alpha \star \psi$,, then

$$\delta_\alpha\tilde{\psi} = \delta_\alpha(\mu^{-1/2}\psi) = i\mu^{-1/2}(\Lambda_\alpha \star \psi) \neq i\Lambda_\alpha \star \tilde{\psi}$$

and the action (4.3) will not be gauge invariant.

Nevertheless, demanding

$$\delta_\alpha\tilde{\psi} = i\Lambda_\alpha \star \tilde{\psi}$$

we can reconstruct the Seiberg-Witten map for the field $\tilde{\psi}$, but this time taking the solution in the $a \to 0$ limit as $\tilde{\psi}^0 = \mu^{-1/2}\psi^0$ instead of ψ^0. This is allowed by the transformation law $\delta_\alpha\tilde{\psi}^0 = i\alpha\tilde{\psi}^0$. Repeating the same calculation we find the following solution:

$$\tilde{\psi} = \mu^{-1/2}\psi^0 - \mu^{-1/2}\frac{1}{2}C^{\rho\sigma}_\lambda x^\lambda A^0_\rho \partial_\sigma\psi^0 - \mu^{-1/2}\frac{na}{4}A^0_n\psi^0\,. \tag{4.7}$$

Having obtained the way to rescale the field $\tilde{\psi}$ and using the solutions for the Seiberg-Witten map, we write down the equations of motion from (4.5) up to first order in a

$$\begin{aligned}
&(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{2}C^{\rho\sigma}_\lambda\gamma_\rho\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0 - \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) - \frac{i}{4}C^{\rho\sigma}_\sigma\gamma^\mu F^0_{\mu\rho}\psi^0 = 0\,,\\
&-i\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\mu - m\bar{\psi}^0 - \frac{1}{2}C^{\rho\sigma}_\lambda\overline{\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0}\gamma_\rho + \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\overline{\mathcal{D}^0_\sigma\psi^0}\gamma^\mu F^0_{\mu\rho} + \frac{i}{4}C^{\rho\sigma}_\sigma\bar{\psi}^0\gamma^\mu F^0_{\mu\rho} = 0\,.
\end{aligned} \tag{4.8}$$

However, we are also interested in the effective action for fermions up to first order in a. Let us write the action (4.3) with all derivatives explicitly, using (3.8) and (3.12):

$$\begin{aligned}
S_m = \int d^{n+1}x\,\mu(x)\Big(&\bar{\tilde{\psi}} \star (i\gamma^\mu\tilde{D}^*_\mu - m)\tilde{\psi} + \bar{\tilde{\psi}} \star \gamma^i V_i \star e^{-ia\partial_n}\tilde{\psi} + \bar{\tilde{\psi}} \star \gamma^n V_{n1} \star e^{-ia\partial_n}\tilde{\psi} + \\
&+ \bar{\tilde{\psi}} \star \gamma^n V_{n2} \star \tilde{D}^*_n\tilde{\psi} + \bar{\tilde{\psi}} \star \gamma^n V^j_{n3} \star \tilde{D}^*_j\tilde{\psi}\Big)\,.
\end{aligned} \tag{4.9}$$

Owing to the cyclicity property of the integral (4.1), we can omit one $\star$ in the above action. Then we rescale the fermionic fields using (4.7) and, finally, we insert the solutions for the Seiberg-Witten map for the gauge field and obtain up to first order in a[8]

$$\begin{aligned}
S_m = \int d^{n+1}x\Big\{&\bar{\psi}^0(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4}C^{\rho\sigma}_\lambda x^\lambda\bar{\psi}^0 F^0_{\rho\sigma}(i\gamma^\mu\mathcal{D}^0_\mu - m)\psi^0 - \\
&- \frac{1}{2}C^{\rho\sigma}_\lambda\bar{\psi}^0\gamma_\rho\mathcal{D}^0_\sigma\mathcal{D}^{0\lambda}\psi^0 - \frac{i}{2}C^{\rho\sigma}_\lambda x^\lambda\bar{\psi}^0\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) - \frac{i}{4}C^{\rho\sigma}_\sigma\bar{\psi}^0\gamma^\mu F^0_{\mu\rho}\psi^0\Big\}\,.
\end{aligned} \tag{4.10}$$

[8] The covariant derivative $\mathcal{D}^0_\mu$ is the usual covariant derivative for the undeformed U(1) gauge field theory, $\mathcal{D}^0_\mu = \partial_\mu - iA^0_\mu$.

Since the integral in (4.10) is the usual integral, applying the variational principle to (4.10) leads to the usual Euler-Lagrange equation of motion[9]

$$\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \psi)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} + \frac{\partial \mathcal{L}}{\partial \psi} = 0\,. \tag{4.11}$$

Using (4.11) the equations of motion for the fields $\bar{\psi}^0$ and ψ^0 follow from (4.10). They are the same as in (4.8), so we do not write them again.

Now we look at the action for the gauge field

$$S_g = -\frac{1}{4}\int d^{n+1}x\ \mu(x)\mathrm{Tr}\,(X_2 \star F_{\mu\nu} \star F^{\mu\nu})\,. \tag{4.12}$$

Here the gauge covariant expression X_2

$$\delta_\alpha X_2 = i[\Lambda_\alpha \stackrel{\star}{,} X_2] \tag{4.13}$$

has been introduced in order to obtain the proper limit $a \to 0$ of the equations of motion, see ref. [12]. From (4.13) we obtain, up to first order in a,

$$X_2 = (1 - anA_n^0)\mu^{-1}\,. \tag{4.14}$$

Expanding (4.12) up to first order in a and using the solutions for the Seiberg-Witten map, we obtain the effective action for the gauge field

$$S_g = -\frac{1}{4}\int d^{n+1}x \left\{ F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{2}C_\lambda^{\rho\sigma}x^\lambda F^{0\mu\nu}F^0_{\mu\nu}F^0_{\rho\sigma} + 2C_\lambda^{\rho\sigma}x^\lambda F^{0\mu\nu}F^0_{\mu\rho}F^0_{\nu\sigma}\right\}. \tag{4.15}$$

The complete action for U(1) gauge theory coupled with matter is $S = S_m + S_g$. The equations of motion for the matter fields are given in equations (4.8). Using the standard Euler-Lagrange equation of motion, for the gauge field we obtain

$$\begin{aligned}
-J^\rho = {} & \partial_\mu F^{0\mu\rho} - \frac{1}{2}C_\mu^{\alpha\beta}F^{0\mu\rho}F^0_{\alpha\beta} - \frac{1}{4}C_\mu^{\mu\rho}F^{0\alpha\beta}F^0_{\alpha\beta} + C_\mu^{\alpha\beta}F^{0\mu}_{\alpha}F^{0\rho}_{\beta} - \\
& - C_\mu^{\alpha\rho}F^{0\beta\mu}F^0_{\alpha\beta} + C_\mu^{\alpha\mu}F^{0\beta\rho}F^0_{\alpha\beta} + \\
& + C_\lambda^{\alpha\beta}x^\lambda\partial_\mu\left(F^{0\mu}_{\alpha}F^{0\rho}_{\beta} - \frac{1}{2}F^{0\mu\rho}F^0_{\alpha\beta}\right) - \frac{1}{4}C_\lambda^{\mu\rho}x^\lambda\partial_\mu(F^{0\alpha\beta}F^0_{\alpha\beta}) - \\
& - C_\lambda^{\alpha\rho}x^\lambda\partial_\mu(F^{0\beta\mu}F^0_{\alpha\beta}) + C_\lambda^{\alpha\mu}x^\lambda\partial_\mu(F^{0\beta\rho}F^0_{\alpha\beta}).
\end{aligned} \tag{4.16}$$

The current J^ρ is given by

$$\begin{aligned}
J^\rho = {} & -\partial_\mu\frac{\partial\mathcal{L}_m}{\partial(\partial_\mu A^0_\rho)} + \frac{\partial\mathcal{L}_m}{\partial A^0_\rho} \\
= {} & \bar{\psi}^0\gamma^\rho\psi^0 - \frac{1}{2}C_\lambda^{\alpha\beta}x^\lambda\bar{\psi}^0\gamma^\rho F^0_{\alpha\beta}\psi^0 - C_\lambda^{\alpha\rho}x^\lambda\bar{\psi}^0\gamma^\mu F^0_{\mu\alpha}\psi^0 - \frac{i}{2}C_\lambda^{\alpha\mu}\eta^{\lambda\rho}\overline{\mathcal{D}^0_\mu\psi^0}\gamma_\alpha\psi^0 - \\
& - \frac{i}{2}C_\mu^{\alpha\rho}\bar{\psi}^0\gamma^\mu(\mathcal{D}^0_\alpha\psi^0) + \frac{i}{2}C_\mu^{\alpha\rho}\bar{\psi}^0\gamma_\alpha(\mathcal{D}^{0\mu}\psi^0) + \frac{i}{4}C_\alpha^{\alpha\mu}\left(\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\rho\psi^0 - \bar{\psi}^0\gamma^\rho\mathcal{D}^0_\mu\psi^0\right) + \\
& + \frac{i}{2}C_\lambda^{\alpha\mu}x^\lambda\overline{\mathcal{D}^0_\mu\psi^0}\gamma^\rho(\mathcal{D}^0_\alpha\psi^0)\,.
\end{aligned} \tag{4.17}$$

[9]Note that in (4.10), the lagrangian density depends on the second derivatives of fields as well.

5. Symmetries and the Seiberg-Witten map

Using the equations of motion (4.8) one can show that the current (4.17) is conserved, $\partial_\mu J^\mu = 0$. In the undeformed gauge theory, existence and conservation of the current J^μ are consequences of the symmetry of the action with respect to gauge transformations. One expects that the same applies here. To check this, we calculate the variation of the action (4.10) when $\delta_\alpha \psi^0 = i\alpha\psi^0$, $\delta_\alpha \bar\psi^0 = -i\alpha\bar\psi^0$ and $\delta_\alpha A^0_\mu = \partial_\mu \alpha$[10]

$$\delta S_m = \int d^{n+1}x \alpha\, \partial_\mu j^\mu = 0. \tag{5.1}$$

Here j^ρ is given by

$$\begin{aligned} j^\rho &= \bar\psi^0\gamma^\rho\psi^0 - \frac{1}{4}C^{\alpha\beta}_\lambda x^\lambda \bar\psi^0\gamma^\rho F^0_{\alpha\beta}\psi^0 - \frac{1}{2}C^{\alpha\rho}_\lambda x^\lambda\bar\psi^0\gamma^\mu F^0_{\mu\alpha}\psi^0 - \\ &- \frac{i}{4}C^{\alpha\mu}_\lambda \eta^{\lambda\rho}\Big(\overline{\mathcal{D}^0_\mu\psi^0}\gamma_\alpha\psi^0 - \bar\psi^0\gamma_\alpha\mathcal{D}^0_\mu\psi^0\Big) - \frac{i}{4}C^{\alpha\rho}_\lambda\Big(\overline{\mathcal{D}^{0\lambda}\psi^0}\gamma_\alpha\psi^0 - \bar\psi^0\gamma_\alpha\mathcal{D}^{0\lambda}\psi^0\Big)\,. \end{aligned} \tag{5.2}$$

Comparing this result with (4.17) there seem to be two different conserved currents in our theory. A difference between these two currents is not topological, nor are the corresponding conserved charges equal. Apparently, we have additional (gauge) symmetry in the model. The source of this symmetry must be the Seiberg-Witten map.

It is well known that the Seiberg-Witten map is not unique [13]. An analysis of the ambiguities in the Seiberg-Witten map in the canonical noncommutative space was provided in ref. [14] and we adapt this analysis to the problem at hand. The important difference with the respect to the canonical case is that we allow derivative-valued gauge fields as solutions of the Seiberg-Witten map. In our setting the derivative-valued gauge fields appear naturally, as a consequence of non-trivial Leibniz rules for the Dirac operator (2.9). Note that we discuss only the ambiguities relevant to the classical action, compare [15].

In ref. [14] it was shown that the possible ambiguities in the Seiberg-Witten map for the gauge parameter did not affect the action. On the other hand, the solution for the Seiberg-Witten map for fermions (4.7) allows an additional term,

$$\Delta\tilde\psi = b_1\mu^{-1/2}C^{\rho\sigma}_\lambda x^\lambda F^0_{\rho\sigma}\psi^0\,, \tag{5.3}$$

which does affect the action.

Furthermore, to the solution for the vector fields (3.10) and (3.16) we can add the following terms:[11]

$$\begin{aligned} \Delta\tilde V_\mu &= ib_2 C^{\rho\sigma}_\lambda x^\lambda F^0_{\mu\rho}\tilde{\mathcal{D}}^0_\sigma + ib_3C^{\rho\sigma}_\lambda x^\lambda F^0_{\rho\sigma}\tilde{\mathcal{D}}^0_\mu + \\ &+ \frac{i}{2}(b_2-2b_3)C^{\rho\sigma}_\lambda x^\lambda(\mathcal{D}^0_\sigma F^0_{\mu\rho}) + \frac{ia}{2}(nb_2-2b_3)F^0_{\mu n}\,, \end{aligned} \tag{5.4}$$

[10] Note that $\delta_\alpha F^0_{\mu\nu} = 0$.

[11] These additional terms are obtained either as a solution of the homogeneous part of equation (3.7), or starting from a more general ansatz for the gauge field (3.8): $\tilde V_j = V_{j1}\star e^{-ia\partial_n} + V^n_{j2}\star\tilde D^*_n + V_3\star\tilde D^*_j + V^l_{j4}\star\tilde D^*_l$

where the coefficients of the last two terms are fixed demanding that $\Delta\tilde{V}_\mu$ should be hermitean. Note also that $\mathcal{D}^0_\sigma F^0_{\mu\rho} = \partial^0_\sigma F^0_{\mu\rho}$ because we work with U(1) gauge theory. This results in the modification of the curvature-like terms

$$\begin{aligned}\Delta F_{\mu\nu} &= 2b_2 C^{\rho\sigma}_\lambda x^\lambda F_{\mu\sigma}F_{\nu\rho} + 2b_3 C^{\rho\sigma}_\lambda x^\lambda F_{\mu\nu}F_{\rho\sigma} - \\ &- \frac{i}{2}(b_2 - 2b_3)C^{\rho\sigma}_\lambda x^\lambda \Big(\mathcal{D}^0_\nu \mathcal{D}^0_\sigma F^0_{\mu\rho} - \mathcal{D}^0_\mu \mathcal{D}^0_\sigma F^0_{\nu\rho}\Big) - \frac{ia}{2}(n-1)b_2 \mathcal{D}^0_n F^0_{\mu\nu} \end{aligned} \tag{5.5}$$

and torsion-like terms

$$\begin{aligned}\Delta T^\sigma_{\mu\nu} &= -ib_2\Big(C^{\rho\sigma}_\nu F^0_{\mu\rho} - C^{\rho\sigma}_\mu F^0_{\nu\rho}\Big) - ib_2 C^{\rho\sigma}_\lambda x^\lambda(\mathcal{D}^0_\nu F^0_{\mu\rho} - \mathcal{D}^0_\mu F^0_{\nu\rho}) - \\ &- ib_3\Big(\delta^\sigma_\mu(C^{\rho\alpha}_\nu F^0_{\rho\alpha} + C^{\rho\alpha}_\lambda x^\lambda(\mathcal{D}^0_\nu F^0_{\rho\alpha})) - \delta^\sigma_\nu(C^{\rho\alpha}_\mu F^0_{\rho\alpha} + C^{\rho\alpha}_\lambda x^\lambda(\mathcal{D}^0_\mu F^0_{\rho\alpha}))\Big)\,,\end{aligned} \tag{5.6}$$

of the field strength (3.21).

Finally it is possible to add the following expression to the solution (4.14) for X_2:

$$\Delta X_2 = b_4 \mu^{-1} C^{\rho\sigma}_\lambda x^\lambda F^0_{\rho\sigma}\,. \tag{5.7}$$

Taking into consideration all the additional terms (5.3)–(5.5) and (5.7), we obtain a more general effective action:

$$\begin{aligned} S = \int d^{n+1}x \Big\{ &\bar\psi^0(i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4}F^0_{\mu\nu}F^{0\mu\nu} - \\ &- \frac{1}{4}C^{\rho\sigma}_\lambda\left(\bar\psi^0\gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda}\psi^0}\gamma_\rho\psi^0\right) - \\ &- \frac{1}{4}(1-8b_1)C^{\rho\sigma}_\lambda x^\lambda \bar\psi^0 F^0_{\rho\sigma}(i\gamma^\mu(\mathcal{D}^0_\mu\psi^0) - m\psi^0) - \\ &- \frac{i}{2}(1-2b_2)C^{\rho\sigma}_\lambda x^\lambda \bar\psi^0\gamma^\mu F^0_{\mu\rho}(\mathcal{D}^0_\sigma\psi^0) + \\ &+ ib_3 C^{\rho\sigma}_\lambda x^\lambda \bar\psi^0\gamma^\mu F^0_{\rho\sigma}(\mathcal{D}^0_\mu\psi^0) - \\ &- 2i(b_1 - \frac{1}{4}(b_2 - 2b_3))C^{\rho\sigma}_\lambda x^\lambda \bar\psi^0\gamma^\mu(\mathcal{D}^0_\sigma F^0_{\mu\rho})\psi^0 - \\ &- \frac{ia}{4}(n + 8b_1 - 2nb_2 + 4b_3 - 1)\bar\psi^0\gamma^\mu F^0_{\mu n}\psi^0 - \\ &- \frac{1}{2}(1-2b_2)C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu}F^0_{\mu\rho}F^0_{\nu\sigma} + \frac{1}{8}(1 - 8b_3 - 2b_4)C^{\rho\sigma}_\lambda x^\lambda F^{0\mu\nu}F^0_{\mu\nu}F^0_{\rho\sigma}\Big\}. \end{aligned} \tag{5.8}$$

All constants b_i are completely undetermined, and were all set to zero in previous calculations. The reason for this particular choice was a technical simplicity in constructing the Seiberg-Witten map. However, we have another interesting possibility. There exist a particular choice of the constants b_i such that all ambiguous, undetermined terms in the action (5.8) are set to zero.

For massless fermions, we choose $b_1 = 1/16, b_2 = 1/2, b_3 = 1/8, b_4 = 0$, and for massive fermions, we choose $b_1 = 1/8, b_2 = 1/2, b_3 = 0, b_4 = 1/2$. The effective action for U(1)

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gauge theory with fermionic matter[12] up to first order in the deformation parameter a is

$$S = \int d^{n+1}x \Big\{ \bar{\psi}^0 (i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - \\ - \frac{1}{4} C^{\rho\sigma}_{\lambda} \left(\bar{\psi}^0 \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi^0 + \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi}^0 \gamma_\rho \psi^0 \right) \Big\} . \qquad (5.9)$$

The corresponding equations of motion are given as

$$\begin{aligned} &(i\gamma^\mu \mathcal{D}^0_\mu - m)\psi^0 - \frac{1}{2} C^{\rho\sigma}_{\lambda} \gamma_\rho \mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi^0 - \frac{i}{4n} C^{\rho\sigma}_{\sigma} \gamma^\mu F^0_{\mu\rho} \psi^0 = 0 , \\ &-i\overline{\mathcal{D}^0_\mu \psi^0} \gamma^\mu - m\bar{\psi}^0 - \frac{1}{2} C^{\rho\sigma}_{\lambda} \overline{\mathcal{D}^0_\sigma \mathcal{D}^{0\lambda} \psi^0} \gamma_\rho + \frac{i}{4n} C^{\rho\sigma}_{\sigma} \bar{\psi}^0 \gamma^\mu F^0_{\mu\rho} = 0 , \\ &\partial_\mu F^{0\rho\mu} = J^\rho \\ &\qquad = \bar{\psi}^0 \gamma^\rho \psi^0 - \frac{i}{4} \Big(C^{\alpha\mu}_{\lambda} \eta^{\lambda\rho} \Big(\overline{\mathcal{D}^0_\mu \psi^0} \gamma_\alpha \psi^0 - \bar{\psi}^0 \gamma_\alpha \mathcal{D}^0_\mu \psi^0 \Big) + \\ &\qquad\qquad + C^{\alpha\rho}_{\lambda} \Big(\overline{\mathcal{D}^{0\lambda} \psi^0} \gamma_\alpha \psi^0 - \bar{\psi}^0 \gamma_\alpha \mathcal{D}^{0\lambda} \psi^0 \Big) \Big) . \end{aligned}$$

Calculating the current j^ρ from the variation of the action (5.1) with the above choice of constants, we obtain $j^\rho = J^\rho$. Furthermore, we obtain the action (5.9) and equations of motion (5.10) that are both gauge invariant and invariant under classical Poincaré transformations!

We end this analysis with a few comments. First, note that $j^\rho = J^\rho$ means that the fermionic action, up to first order in the deformation parameter, is

$$S_m = \frac{1}{2} \int d^{n+1}x \left\{ \bar{\psi}^0 (i\gamma^\mu \mathcal{D}_\mu - m)\psi^0 + (-i\overline{\mathcal{D}_\mu \psi} \gamma^\mu - m\bar{\psi})\psi \right\} , \qquad (5.10)$$

where the operator $\mathcal{D}_\mu$ is the Dirac operator (2.6) expanded in a in which partial derivatives are covariantised by the minimal substitution $\partial_\mu \to \partial_\mu - iA^0_\mu$. We conjecture that this fact might be valid to all orders, but one needs to be careful in ordering derivatives in the expansion of the Dirac operator.

Note also that if one allows for the derivative-valued gauge fields in the canonical case, one can construct an effective U(1) action with no additional terms in the first order with respect to the undeformed U(1) gauge theory[13] (compare with ref. [17]). For the classical theory, one can use the Seiberg-Witten map (gauge freedom) to transform additional gauge interactions into the geometry of spacetime (the torsion-like part of the field strength). Unfortunately, we still do not have a clear understanding of the interplay between deformed symetries, gauge theories and spacetime geometry. As a next step, an investigation of deformed general coordinate transformations is performed in ref. [18].

Acknowledgments

We thank Frank Meyer and Julius Wess for their help and many useful discussions.

[12] Of course, $m = 0$ in the massless case.

[13] This has already been known to L.M., see ref. [16].

L.J. gratefully acknowledges the support of the Ministry of Science and Technology of the Republic of Croatia under the contract 0098003 and partial support of the Alexander von Humboldt Foundation.

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